Utility Functions
Introduction To Utilitiy Functions

Gary Schurman MBE, CFA

October 2023

Utility functions give us a way to measure investor's preferences for wealth and the amount of risk they are willing to undertake in the hope of attaining greater wealth. This makes it possible to develop a theory of portfolio optimization. Thus utility theory lies at the heart of modern portfolio theory. Note that the numerical values that a utility function generates are used for comparison only (ordinal values are relevant, cardinal values are not).

Imagine that we are presented with an investment opportunity that promises to increase our wealth by W_i given the i'th state of the world, which is a random variable. We will define the variable W_i to be the increase in wealth given the i'th state of the world, the function $U(W_i)$ to be the utility from this increase in wealth, the variable p_i to be the probability of realizing the i'th state of the world, and the variable n to be the number of possible states of the world. The equations for utility and expected utility are...

$$\text{Utility of wealth} = U(W_i) \ \text{...such that...}\ \text{Expected utility} = \mathbb{E}\Big[U(W)\Big] = \sum_{i=1}^{n} p_i\,U(W_i) \tag{1}$$

Note that the derivatives of the utility function in Equation (1) above imply the following...

First derivative: $U'(W_i) =$ Marginal utility of an additional dollar of wealth given the current level of weath
Second derivative: $U''(W_i) =$ Change in marginal utility given the current level of weath

We will examine the following generic utility functions...

Description	Equation	Notes
Linear utility	$U(W_i) = 1 + \alpha\,W_i$	Investor is risk-neutral
Logrithmic utility	$U(W_i) = \ln(1 + \alpha\,W_i)$	Investor is risk-averse
Exponential utility	$U(W_i) = \text{Exp}\left\{1 + \alpha\,W_i\right\}$	Investor is risk-averse
Quadratic utility	$U(W_i) = W_i - \alpha\,W_i^2$	Investor is risk-averse

We will define the variable λ to be the Arrow-Pratt measure of risk aversion. Using the derivative equations above, the equation for this measre of risk aversion is...

$$\lambda = -\frac{\text{Second derivative}}{\text{First derivative}} = -\frac{U''(W_i)}{U'(W_i)} \tag{2}$$

Note that $\lambda = 0$ implies risk neutrality, $\lambda > 0$ implies risk aversion.

The certainty equivalent (CE) is the guaranteed investment return that an investor would accept rather than taking a chance on a higher, but uncertain, risky investment return. The certainty equivalent is the guaranteed amount of cash that an investor would consider as having the same amount of desirability as a risky asset. Using Equation (1) above, the certainty equivalent is the following equality solved for CE...

$$U(CE) = \mathbb{E}\Big[U(W)\Big] \tag{3}$$

We will define the variable μ to be the risk-free rate. Given that the certainty equivalent CE is a sum certain to be received at time T, the value of the certainty eqivalent at time zero is the certainty equivalent discounted at the risk-free rate over the time interval $[0, T]$. Using Equation (3) above, the equation for the present value of the certainty equivalent is...

$$\text{Present value of the certainty equivalent} = PVCE = CE\,(1+\mu)^{-T} \tag{4}$$

We will define the variable κ to be the risk-adjusted discount rate, which should equate to the investment's rate of return over the time interval $[0, T]$. Using Equations (1) and (4) above, the equation for the risk-adjusted discount rate is...

$$\kappa = \left(\mathbb{E}\Big[W\Big]\Big/PVCE\right)^{1/T} - 1 \tag{5}$$

If we are given the discount rate κ then using Equations (4) and (5) above the equation for the value of the certainty equivalent is...

$$CE = \mathbb{E}\Big[W\Big]\left(\frac{1+\mu}{1+\kappa}\right)^{T} \tag{6}$$